

CORON PROBLEM FOR FRACTIONAL EQUATIONS

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ABSTRACT. We prove that the critical problem for the fractional Laplacian in an annular type domain admits a positive solution provided that the inner hole is sufficiently small.

1. INTRODUCTION

Let $N \geq 3$ and Ω be a smooth bounded domain of \mathbb{R}^N . The classical formulation of Coron problem goes back to 1984 and says that if there is a point $x_0 \in \mathbb{R}^N$ and radii $R_2 > R_1 > 0$ such that

$$(1.1) \quad \{R_1 \leq |x - x_0| \leq R_2\} \subset \Omega, \quad \{|x - x_0| \leq R_1\} \not\subset \overline{\Omega},$$

then the critical elliptic problem

$$(1.2) \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a solution provided that R_2/R_1 is sufficiently large [6]. A few years later Bahri and Coron [1], in a seminal paper, considerably improved this existence result by showing, via sophisticated topological arguments based upon homology theory, that (1.2) admits a solution provided that $H_m(\Omega, \mathbb{Z}_2) \neq \{0\}$ for some $m > 0$. Furthermore, in [8, 11, 14] the authors show that existence of a solution is possible also in some contractible domains. Let $N \geq 2$ and $s \in (0, 1)$ with $N > 2s$, and consider the nonlocal fractional problem

$$(1.3) \quad \begin{cases} (-\Delta)^s u = u^{\frac{N+2s}{N-2s}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

involving the fractional Laplacian $(-\Delta)^s$. Here, for smooth functions φ , $(-\Delta)^s \varphi$ is defined by

$$(-\Delta)^s \varphi(x) = C(N, s) \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where

$$(1.4) \quad C(N, s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+2s}} d\zeta \right)^{-1};$$

see [10]. Fractional Sobolev spaces are introduced in the middle part of the last century, especially in the framework of harmonic analysis. More recently, after the paper of Caffarelli and Silvestre [2], a large amount of papers were written on problems which involve the fractional diffusion $(-\Delta)^s$, $0 < s < 1$. Due to its nonlocal character, working on bounded domains imposes that an appropriate variational formulation of the problem is to consider functions on \mathbb{R}^N with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ replacing the boundary condition $u = 0$ on $\partial\Omega$. We set $X_0 = \{u \in \dot{H}^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$ and we consider the formulation

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx = \int_{\Omega} u^{\frac{N+2s}{N-2s}} \varphi dx \quad \text{for all } \varphi \in X_0.$$

It has been proved recently [15, Corollary 1.3] that if Ω is a star-shaped domain, then problem (1.3) does not admit solutions and that for exponents larger than $(N + 2s)/(N - 2s)$ the problem does not admit any nontrivial solution thus dropping the positivity requirement. It is then natural to think that, as in

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the local case $s = 1$, by assuming suitable geometrical or topological conditions on Ω one can get the existence of nontrivial solutions. We note that Capella [3] studies the problem for the particular case $s = 1/2$ by using the Caffarelli reduction to transform the problem in a local form and that Servadei and Valdinoci [16] studies the Brezis-Nirenberg problem with the fractional Laplacian.

The main result of the paper is the following Coron type result in the fractional setting.

Theorem 1.1. *If (1.1) holds, then (1.3) admits a weak solution in X_0 for R_2/R_1 sufficiently large.*

We roughly recall Coron's argument [6] for the case $s = 1$. Although the corresponding Rayleigh quotient does not attain the infimum value, say \mathbb{S} , the global compactness theorem due to Struwe [17] implies that it satisfies the Palais-Smale condition at each level in $(\mathbb{S}, 2^{2/N}\mathbb{S})$. He introduced a test function defined on a small ball which contains the small hole of Ω , and he showed that under assumption (1.1), the maximum value of the test function is less than $2^{2/N}\mathbb{S}$. If there is no critical point of the Rayleigh quotient in $(\mathbb{S}, 2^{2/N}\mathbb{S})$, he showed that the small ball can be retracted into its boundary, which is a contradiction. For the case $s \in (0, 1)$, one of the main difficulties that one has to face is to get a uniform estimate for the energy of truncations of the family of functions

$$(1.5) \quad U_{\varepsilon, z}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - z|^2} \right)^{\frac{N-2s}{2}}, \quad z \in \mathbb{R}^N, \varepsilon > 0,$$

which are precisely obtained in Propositions 2.1-2.2. We note that $U_{\varepsilon, z}$ satisfies $(-\Delta)^s u = u^{(N+2s)/(N-2s)}$ in \mathbb{R}^N up to a constant, and $U_{\varepsilon, z}$ with the constant factor is called Talenti function for the fractional Laplacian. The other difficulty for the case $s \in (0, 1)$ is global compactness. We give a compactness result which is sufficient for our arguments.

2. PRELIMINARY RESULTS

We define

$$\dot{H}^s(\mathbb{R}^N) = \{u \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N) : \|u\| < \infty\},$$

where

$$\|u\| = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

We also define

$$\langle u, v \rangle = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \quad \text{for each } u, v \in \dot{H}^s(\mathbb{R}^N).$$

Then we know that $\dot{H}^s(\mathbb{R}^N)$ is a Hilbert space with the inner product above, it is continuously embedded into $L^{2N/(N-2s)}(\mathbb{R}^N)$ and it holds

$$(2.1) \quad \langle u, v \rangle = \frac{2}{C(N, s)} \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx \quad \text{for each } u, v \in \dot{H}^s(\mathbb{R}^N),$$

where $C(N, s)$ is the constant given in (1.4); see [10]. We set

$$X_0 = \{u \in \dot{H}^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Since it is a closed subspace of $\dot{H}^s(\mathbb{R}^N)$, X_0 itself is also a Hilbert space, and we use the same symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for its inner product and norm. We note

$$\|u\| = \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2} \quad \text{for each } u \in X_0,$$

where $Q = \mathbb{R}^{2N} \setminus (\mathbb{C}\Omega \times \mathbb{C}\Omega)$.

Since we have (2.1), for the sake of simplicity, we will find a positive weak solution of

$$(2.2) \quad \begin{cases} (-\Delta)^s u = \frac{C(N, s)}{2} |u|^{\frac{4s}{N-2s}} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

which is equivalent to find a weak solution of (1.3). Here, we say $u \in X_0$ is a weak solution to (2.2) if it satisfies

$$\iint_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} |u|^{\frac{4s}{N-2s}} u \varphi dx \quad \text{for each } \varphi \in X_0.$$

Without loss of generality, we may assume (1.1) with $x_0 = 0 \notin \bar{\Omega}$, R_2 is fixed with $R_2 > 10$ and $R_1 = \delta \in (0, 1/20]$ which will be fixed later. We set $B_r = \{x \in \mathbb{R}^N : |x| \leq r\}$ for $r > 0$. Without loss

of generality, we may also assume $\Omega \cap B_\delta = \emptyset$ and $B_5 \setminus B_{3\delta/2} \subset \Omega$. Let $\varphi_\delta : \mathbb{R}^N \rightarrow [0, 1]$ be a smooth radially symmetric function such that

$$\varphi_\delta(x) = \begin{cases} 0 & \text{if } 0 \leq |x| \leq 2\delta \text{ and } |x| \geq 4, \\ 1 & \text{if } 4\delta \leq |x| \leq 3, \end{cases}$$

$$|\nabla \varphi_\delta(x)| \leq \delta^{-1}, \quad \text{for } x \in B_{4\delta}, \quad |\nabla \varphi_\delta(x)| \leq 2, \quad \text{for } x \in \mathbb{C}B_3.$$

For $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$, we set

$$u_{\delta, \varepsilon, z}(x) = \varphi_\delta(x) U_{\varepsilon, z}(x),$$

where the $U_{\varepsilon, z}$ were defined in (1.5). The next estimates will be crucial for the proof of Theorem 1.1.

Proposition 2.1. *There exists $C_1 > 0$ such that*

$$(2.3) \quad \|u_{\delta, \varepsilon, z}\|^2 \leq \|U_{\varepsilon, z}\|^2 + C_1 \left(\left(\frac{\delta}{\varepsilon} \right)^{N-2s} + \left(\frac{\delta}{\varepsilon} \right)^{N+2-2s} + \varepsilon^{N-2s} \right)$$

for each $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$, and

$$(2.4) \quad \|u_{\delta, \varepsilon, z}\|^2 \leq \|U_{\varepsilon, z}\|^2 + C_1 \varepsilon^{N-2s} (1 + \delta^{-2s})$$

for each $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1 \setminus B_{1/2}$.

Proof. Let $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$. We define

$$D = \{(x, y) \in (B_4 \times \mathbb{C}B_3) \cup (\mathbb{C}B_3 \times B_4) : |x - y| > 1\},$$

$$E = \{(x, y) \in (B_4 \times \mathbb{C}B_3) \cup (\mathbb{C}B_3 \times B_4) : |x - y| \leq 1\},$$

$$\tilde{D} = \{(x, y) \in (B_{4\delta} \times (B_4 \setminus B_{4\delta})) \cup ((B_4 \setminus B_{4\delta}) \times B_{4\delta}) : |x - y| > \delta\}$$

and

$$\tilde{E} = (B_{4\delta} \times B_{4\delta}) \cup \{(x, y) \in (B_{4\delta} \times (B_4 \setminus B_{4\delta})) \cup ((B_4 \setminus B_{4\delta}) \times B_{4\delta}) : |x - y| \leq \delta\}.$$

Then we have

$$\mathbb{R}^{2N} = \tilde{E} \cup \tilde{D} \cup E \cup D \cup ((B_3 \setminus B_{4\delta}) \times (B_3 \setminus B_{4\delta})) \cup (\mathbb{C}B_4 \times \mathbb{C}B_4).$$

We remark that this is *not* a disjoint union. We can easily see that

$$\int_{(B_3 \setminus B_{4\delta}) \times (B_3 \setminus B_{4\delta})} \left(\frac{|u_{\delta, \varepsilon, z}(x) - u_{\delta, \varepsilon, z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon, z}(x) - U_{\varepsilon, z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy = 0$$

and

$$\int_{\mathbb{C}B_4 \times \mathbb{C}B_4} \left(\frac{|u_{\delta, \varepsilon, z}(x) - u_{\delta, \varepsilon, z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon, z}(x) - U_{\varepsilon, z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \leq 0.$$

We shall denote by C generic positive constants, possibly varying from line to line, and which do not depend on $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$. For each $(x, y) \in \mathbb{R}^{2N}$, we have

$$(2.5) \quad \begin{aligned} & \frac{|u_{\delta, \varepsilon, z}(x) - u_{\delta, \varepsilon, z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon, z}(x) - U_{\varepsilon, z}(y)|^2}{|x - y|^{N+2s}} \\ &= \frac{(u_{\delta, \varepsilon, z}(x) + U_{\varepsilon, z}(x) - u_{\delta, \varepsilon, z}(y) - U_{\varepsilon, z}(y))(u_{\delta, \varepsilon, z}(x) - U_{\varepsilon, z}(x) + U_{\varepsilon, z}(y) - u_{\delta, \varepsilon, z}(y))}{|x - y|^{N+2s}} \\ &\leq \frac{2U_{\varepsilon, z}(x)U_{\varepsilon, z}(y)}{|x - y|^{N+2s}}. \end{aligned}$$

From $z \in B_1$, we have

$$\begin{aligned} & \int_{B_4 \times \mathbb{C}B_3, |x-y|>1} \left(\frac{|u_{\delta, \varepsilon, z}(x) - u_{\delta, \varepsilon, z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon, z}(x) - U_{\varepsilon, z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \\ &\leq \int_{B_4 \times \mathbb{C}B_3, |x-y|>1} \frac{2U_{\varepsilon, z}(x)U_{\varepsilon, z}(y)}{|x - y|^{N+2s}} dx dy \leq C\varepsilon^{\frac{N-2s}{2}} \int_{B_4 \times \mathbb{C}B_3, |x-y|>1} \frac{\left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2} \right)^{\frac{N-2s}{2}}}{|x - y|^{N+2s}} dx dy \\ &\leq C\varepsilon^{N-2s} \int_{|\xi| \leq 5} \frac{d\xi}{(\varepsilon^2 + |\xi|^2)^{\frac{N-2s}{2}}} \int_{|\eta| > 1} \frac{d\eta}{|\eta|^{N+2s}} = C\varepsilon^{N-2s} \varepsilon^{2s} \int_{|\xi| \leq 5/\varepsilon} \frac{d\xi}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} \leq C\varepsilon^{N-2s}. \end{aligned}$$

So we can infer

$$\int_D \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \leq C\varepsilon^{N-2s}.$$

We note

$$\nabla U_{\varepsilon,z}(x) = -(N-2s) \left(\frac{\varepsilon}{\varepsilon^2 + |x - z|^2} \right)^{\frac{N-2s}{2}} \frac{x - z}{\varepsilon^2 + |x - z|^2},$$

and

$$\frac{|x - z|}{\varepsilon^2 + |x - z|^2} \leq \frac{1}{2\varepsilon}.$$

Since $|\nabla\varphi_\delta(x)| \leq 2$ for $|x| \geq 4\delta$, $z \in B_1$ and $|tx + (1-t)y| \geq 2$ for each $(x, y) \in E$ and $t \in [0, 1]$,

$$\begin{aligned} & \int_E \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \\ & \leq \int_E \frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} dx dy = \int_E \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx + (1-t)y) \cdot (x - y) dt|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \int_E \frac{\int_0^1 (8|U_{\varepsilon,z}(tx + (1-t)y)|^2 + 2|(\nabla U_{\varepsilon,z})(tx + (1-t)y)|^2) dt}{|x - y|^{N+2s-2}} dx dy \\ & \leq C\varepsilon^{N-2s} \int_E \frac{dx dy}{|x - y|^{N+2s-2}} \\ & \leq C\varepsilon^{N-2s} \int_{|\xi| \leq 4} d\xi \int_{|\eta| \leq 1} \frac{d\eta}{|\eta|^{N+2s-2}} = C\varepsilon^{N-2s}. \end{aligned}$$

From (2.5), we also have

$$\begin{aligned} & \int_{\tilde{D}} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \\ & \leq 2 \int_{\tilde{D}} \frac{U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x - y|^{N+2s}} dx dy = \frac{2}{\varepsilon^{N-2s}} \int_{\tilde{D}} \frac{dx dy}{|x - y|^{N+2s}} \\ & \leq \frac{C}{\varepsilon^{N-2s}} \int_{|\xi| \leq 4\delta} d\xi \int_{|\eta| > \delta} \frac{d\eta}{|\eta|^{N+2s}} \leq C \left(\frac{\delta}{\varepsilon} \right)^{N-2s}. \end{aligned}$$

Since $|\nabla\varphi_\delta(x)| \leq 1/\delta$ for $x \in B_{4\delta}$, we have

$$\begin{aligned} & \int_{\tilde{E}} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} \right) dx dy \\ & \leq \int_{\tilde{E}} \frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x - y|^{N+2s}} dx dy \\ & = \int_{\tilde{E}} \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx + (1-t)y) \cdot (x - y) dt|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \int_{\tilde{E}} \frac{\int_0^1 ((1/\delta)^2 |U_{\varepsilon,z}(tx + (1-t)y)|^2 + |(\nabla U_{\varepsilon,z})(tx + (1-t)y)|^2) dt}{|x - y|^{N+2s-2}} dx dy \\ & \leq C \left(\frac{1}{\delta^2 \varepsilon^{N-2s}} + \frac{1}{\varepsilon^{N-2s}} \cdot \frac{1}{\varepsilon^2} \right) \int_{\tilde{E}} \frac{dx dy}{|x - y|^{N+2s-2}} \\ & \leq C \left(\frac{1}{\delta^2 \varepsilon^{N-2s}} + \frac{1}{\varepsilon^{N+2-2s}} \right) \int_{|\xi| \leq 4\delta} d\xi \int_{|\eta| \leq \delta} \frac{d\eta}{|\eta|^{N+2s-2}} \\ & = C \left(\left(\frac{\delta}{\varepsilon} \right)^{N-2s} + \left(\frac{\delta}{\varepsilon} \right)^{N+2-2s} \right). \end{aligned}$$

By the inequalities above, we obtain (2.3). Let $z \in B_1 \setminus B_{1/2}$. In order to obtain (2.4) we need to consider the integrals on \tilde{D} and \tilde{E} . We have

$$\begin{aligned}
& \int_{(B_4 \setminus B_{4\delta}) \times B_{4\delta}, |x-y| > \delta} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dx dy \\
& \leq \int_{(B_4 \setminus B_{4\delta}) \times B_{4\delta}, |x-y| > \delta} \frac{2U_{\varepsilon,z}(x)U_{\varepsilon,z}(y)}{|x-y|^{N+2s}} dx dy \leq C\varepsilon^{\frac{N-2s}{2}} \int_{(B_4 \setminus B_{4\delta}) \times B_{4\delta}, |x-y| > \delta} \frac{\left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N-2s}{2}}}{|x-y|^{N+2s}} dx dy \\
& \leq C\varepsilon^{N-2s} \int_{|\xi| \leq 5} \frac{d\xi}{(\varepsilon^2 + |\xi|^2)^{\frac{N-2s}{2}}} \int_{|\eta| > \delta} \frac{d\eta}{|\eta|^{N+2s}} \\
& = C\varepsilon^{N-2s} \delta^{-2s} \cdot \varepsilon^{2s} \int_{|\xi| \leq 5/\varepsilon} \frac{d\xi}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} = C\varepsilon^{N-2s} \delta^{-2s}.
\end{aligned}$$

Hence

$$\int_{\tilde{D}} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dx dy \leq C\varepsilon^{N-2s} \delta^{-2s}.$$

Since $|\nabla \varphi_\delta(x)| \leq 1/\delta$ for $x \in \mathbb{R}^N$, $z \in B_1 \setminus B_{1/2}$ and $|tx + (1-t)y| \leq 5\delta \leq 1/4$ for each $(x, y) \in \tilde{E}$ and $t \in [0, 1]$, we have

$$\begin{aligned}
& \int_{\tilde{E}} \left(\frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} - \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} \right) dx dy \\
& \leq \int_{\tilde{E}} \frac{|u_{\delta,\varepsilon,z}(x) - u_{\delta,\varepsilon,z}(y)|^2}{|x-y|^{N+2s}} dx dy \\
& = \int_{\tilde{E}} \frac{|\int_0^1 (\nabla u_{\delta,\varepsilon,z})(tx + (1-t)y) \cdot (x-y) dt|^2}{|x-y|^{N+2s}} dx dy \\
& \leq \int_{\tilde{E}} \frac{\int_0^1 ((1/\delta)^2 |U_{\varepsilon,z}(tx + (1-t)y)|^2 + |(\nabla U_{\varepsilon,z})(tx + (1-t)y)|^2) dt}{|x-y|^{N+2s-2}} dx dy \\
& \leq C\varepsilon^{N-2s} \delta^{-2} \int_{\tilde{E}} \frac{dx dy}{|x-y|^{N+2s-2}} \\
& \leq C\varepsilon^{N-2s} \delta^{-2} \int_{|\xi| \leq 4\delta} d\xi \int_{|\eta| \leq \delta} \frac{d\eta}{|\eta|^{N+2s-2}} = C(\varepsilon\delta)^{N-2s} \leq C\varepsilon^{N-2s}.
\end{aligned}$$

Thus, we obtain the second desired inequality. \square

Proposition 2.2. *There exists $C_2 > 0$ such that*

$$(2.6) \quad \int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}} dx \geq \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}} dx - C_2 \left(\left(\frac{\delta}{\varepsilon} \right)^N + \varepsilon^N \right)$$

for each $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$, and

$$(2.7) \quad \int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}} dx \geq \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}} dx - C_2 \varepsilon^N$$

for each $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1 \setminus B_{1/2}$.

Proof. Let $\delta, \varepsilon \in (0, 1/20]$ and $z \in B_1$. We have

$$\begin{aligned}
& \int_{\mathbb{R}^N} |U_{\varepsilon,z}|^{\frac{2N}{N-2s}} dx - \int_{\mathbb{R}^N} |u_{\delta,\varepsilon,z}|^{\frac{2N}{N-2s}} dx \\
& \leq \int_{|x| \leq 4\delta} \left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2} \right)^N dx + \int_{|x| \geq 3} \left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2} \right)^N dx \leq C \left(\frac{\delta}{\varepsilon} \right)^N + C\varepsilon^N,
\end{aligned}$$

which yields (2.6). By a similar calculation, we can obtain (2.7) as well. \square

Let $I : \dot{H}^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ be given by

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{N-2s}{2N} \int_{\mathbb{R}^N} |u|^{2N/(N-2s)} dx \quad \text{for } u \in \dot{H}^s(\mathbb{R}^N),$$

and let $I_0 : X_0 \rightarrow \mathbb{R}$ be its restriction to X_0 , i.e.,

$$I_0(u) = I(u) \quad \text{for } u \in X_0.$$

Next, let us define $\mathcal{R} : \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \rightarrow \mathbb{R}$ by

$$\mathcal{R}(u) = \frac{\|u\|^2}{\mathcal{N}(u)},$$

where

$$(2.8) \quad \mathcal{N}(u) = \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}}.$$

We also define \mathcal{N}_0 and \mathcal{R}_0 by the restrictions of \mathcal{N} and \mathcal{R} to $X_0 \setminus \{0\}$, respectively. That is,

$$\mathcal{N}_0(u) = \mathcal{N}(u) \quad \text{and} \quad \mathcal{R}_0(u) = \mathcal{R}(u) \quad \text{for } u \in X_0 \setminus \{0\}.$$

Lemma 2.3. $\mathcal{R}_0 \in C^1(X_0 \setminus \{0\})$, and if $\mathcal{R}'_0(v) = 0$ with $v \in X_0$, then $I'_0(\lambda v) = 0$ with some $\lambda > 0$.

Proof. We can easily see $\mathcal{R}_0 \in C^1(X_0 \setminus \{0\})$. Let $v \in X_0$. Since we have

$$\mathcal{R}'_0(v)(\varphi) = \frac{2\mathcal{N}_0(v) \iint_Q \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx dy - 2\|v\|^2 \mathcal{N}_0(v)^{-\frac{2s}{N-2s}} \int_{\Omega} |v|^{\frac{4s}{N-2s}} v \varphi dx}{\mathcal{N}_0(v)^2}$$

for every $\varphi \in X_0$, we have $\mathcal{R}'_0(v) = 0$ if and only if, for every $\varphi \in X_0$,

$$\iint_Q \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx dy = \frac{\|v\|^2}{\int_{\Omega} |v|^{\frac{2N}{N-2s}} dx} \int_{\Omega} |v|^{\frac{4s}{N-2s}} v \varphi dx.$$

Setting λ by

$$(2.9) \quad \lambda^{\frac{4s}{N-2s}} = \frac{\|v\|^2}{\int_{\Omega} |v|^{\frac{2N}{N-2s}} dx},$$

we have $I_0(\lambda v) = 0$. This concludes the proof. \square

We define a manifold of codimension one by setting

$$(2.10) \quad \mathcal{M} = \left\{ u \in X_0 : \int_{\Omega} |u|^{\frac{2N}{N-2s}} dx = 1 \right\}.$$

Lemma 2.4. Let $\{v_n\}_n \subset \mathcal{M}$ be a Palais-Smale sequence for \mathcal{R}_0 at level c . Then

$$u_n = \lambda_n v_n, \quad \lambda_n = \mathcal{R}_0(v_n)^{(N-2s)/(4s)}$$

is a Palais-Smale sequence for I_0 at level $(s/N)c^{N/(2s)}$.

Proof. By following the computations of Lemma 2.3, if λ_n is defined as in (2.9), we have

$$\frac{1}{2} \mathcal{R}'_0(v_n)(\varphi) = \iint_Q \frac{(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx dy - \lambda_n^{\frac{4s}{N-2s}} \int_{\Omega} |v_n|^{\frac{4s}{N-2s}} v_n \varphi dx$$

for every $\varphi \in X_0$. Hence, in turn, by multiplying this identity by λ_n , we conclude that

$$I'_0(u_n)(\varphi) = \iint_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx dy - \int_{\Omega} |u_n|^{\frac{4s}{N-2s}} u_n \varphi dx$$

for every $\varphi \in X_0$. Recalling (2.8) and (2.9), we have

$$\lambda_n = \|v_n\|^{\frac{N-2s}{2s}} = \mathcal{R}_0(v_n)^{\frac{N-2s}{4s}}.$$

From $\mathcal{R}_0(v_n) = c + o(1)$ and $\{v_n\}_n \subset \mathcal{M}$, $\{v_n\}_n$ is bounded in X_0 and so is $\{\lambda_n\}_n$. In particular, it follows that $I'_0(u_n) \rightarrow 0$ in X'_0 as $n \rightarrow \infty$. Moreover, $\{u_n\}_n$ is bounded in X_0 as well, yielding

$$o(1) = I'_0(u_n)(u_n) = \|u_n\|^2 - \int_{\Omega} |u_n|^{\frac{2N}{N-2s}} dx.$$

These facts imply that

$$\lim_{n \rightarrow \infty} I_0(u_n) = \frac{s}{N} \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{\frac{2N}{N-2s}} dx = \frac{s}{N} \lim_{n \rightarrow \infty} \lambda_n^{\frac{2N}{N-2s}} = \frac{s}{N} \left(\lim_{n \rightarrow \infty} \mathcal{R}_0(v_n)^{\frac{N-2s}{4s}} \right)^{\frac{2N}{N-2s}} = \frac{s}{N} c^{N/(2s)},$$

concluding the proof. \square

Let us set

$$\mathbb{S} = \inf\{\mathcal{R}(u) : u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}\}.$$

By [7], we know that

$$\mathcal{R}(U_{\varepsilon,z}) = \mathbb{S} \quad \text{for each } \varepsilon > 0 \text{ and } z \in \mathbb{R}^N,$$

only these functions with any nonzero constant factor attain the infimum,

$$\mathbb{S} = \inf\{\mathcal{R}_0(u) : u \in X_0 \setminus \{0\}\},$$

and the infimum is never attained in the latter case. We also have the following result for sign-changing weak solutions.

Lemma 2.5. *Let $u \in X_0$ be a sign-changing weak solution to (2.2), then $\|u\|^2 \geq 2\mathbb{S}^{N/(2s)}$. Moreover, the same conclusion holds for sign-changing critical points of I .*

Proof. We have $u^\pm \in X_0 \setminus \{0\}$ and

$$|u(x) - u(y)|^2 = |u^+(x) - u^+(y)|^2 + |u^-(x) - u^-(y)|^2 + 2u^+(y)u^-(x) + 2u^+(x)u^-(y)$$

for every $x, y \in \mathbb{R}^N$, where $u^-(x) = -\min\{u(x), 0\}$. This, in turn, implies

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2 + 4 \iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy.$$

By multiplying equation (2.2) by u^\pm easily yields

$$\begin{aligned} \|u^+\|^2 + 2 \iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy &= \int_\Omega |u^+|^{\frac{2N}{N-2s}} dx, \\ \|u^-\|^2 + 2 \iint_Q \frac{u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy &= \int_\Omega |u^-|^{\frac{2N}{N-2s}} dx. \end{aligned}$$

Combining these equalities with $\mathbb{S}\|u^\pm\|_{L^{2N/(N-2s)}}^2 \leq \|u^\pm\|^2$, yields $\int_\Omega |u^\pm|^{2N/(N-2s)} \geq \mathbb{S}^{N/(2s)}$, concluding the proof. \square

Now, we show the following compactness result. In order to show it, we follow the arguments in [18, Section 8.3], which treat the case $s = 1$.

Proposition 2.6. *Let $\{u_n\}_n \subset X_0$ be a Palais-Smale sequence for I_0 at level c with*

$$\frac{s}{N}\mathbb{S}^{N/(2s)} \leq c < \frac{2s}{N}\mathbb{S}^{N/(2s)}.$$

If $(s/N)\mathbb{S}^{N/(2s)} < c < (2s/N)\mathbb{S}^{N/(2s)}$, then $\{u_n\}_n$ converges strongly to a nontrivial constant-sign weak solution to problem (2.2) up to a subsequence, and if $c = (s/N)\mathbb{S}^{N/(2s)}$, then there exist a nontrivial constant-sign weak solution $v \in \dot{H}^s(\mathbb{R}^N)$ to problem

$$(2.11) \quad (-\Delta)^s v = \frac{C(N, s)}{2} |v|^{\frac{4s}{N-2s}} v \quad \text{in } \mathbb{R}^N,$$

to 0 in $\dot{H}^s(\mathbb{R}^N)$ up to a subsequence.

Proof. First, we note that $\{u_n\}_n$ is bounded and $I_0(u_n) = (s/N)\|u_n\|^2 + o(1)$. We may assume that $\{u_n\}_n$ converges weakly to u in X_0 . Then u is a possibly trivial solution to (2.2) and

$$\|u\|^2 \leq \liminf_{n \rightarrow \infty} \|u_n\|^2 = \frac{N}{s} \lim_{n \rightarrow \infty} I_0(u_n) < 2\mathbb{S}^{N/(2s)}.$$

From Lemma 2.5, u is not sign-changing. By a similar argument as in [18, Lemma 8.10], we have

$$I'(u_n - u) \rightarrow 0, \quad I(u_n - u) \rightarrow c - I_0(u) \quad \text{and} \quad \|u_n - u\|^2 \rightarrow \frac{Nc}{s} - \|u\|^2.$$

If $\|u_n - u\|_{L^{2N/(N-2s)}} \rightarrow 0$, we can infer that $\|u_n - u\| \rightarrow 0$, $(s/N)\mathbb{S}^{N/(2s)} < c < (2s/N)\mathbb{S}^{N/(2s)}$ and u is a nontrivial constant-sign solution to (2.2). From here, we consider the case $\|u_n - u\|_{L^{2N/(N-2s)}} \not\rightarrow 0$. Taking small $\delta > 0$, we may assume that $\int_{\mathbb{R}^N} |u_n - u|^{2N/(N-2s)} dx \geq \delta$, for each $n \in \mathbb{N}$. As in the proof of [18, 2) and 3) of Theorem 8.13], we can choose appropriate sequences $\{x_n\}_n \subset \Omega$ and $\{r_n\}_n \subset (0, \infty)$ such that the sequence $\{v_n\}_n \subset \dot{H}^s(\mathbb{R}^N)$ defined by

$$v_n(x) = r_n^{(N-2s)/2} (u_n - u)(r_n x + x_n)$$

converges weakly to $v \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$. We have

$$(2.12) \quad \|v\|^2 \leq \liminf_{n \rightarrow \infty} \|v_n\|^2 = \lim_{n \rightarrow \infty} \|u_n - u\|^2 = \frac{Nc}{s} - \|u\|^2 < 2\mathbb{S}^{N/(2s)} - \|u\|^2.$$

By the boundedness of Ω and $v \neq 0$, we may assume $r_n \rightarrow 0$ and $x_n \rightarrow x_0 \in \bar{\Omega}$. We may also assume that $\{\text{dist}(x_n, \partial\Omega)/r_n\}_n$ has a limit value in $[0, \infty]$. Assume that this limit value is finite. Then v is a solution to the problem

$$(-\Delta)^s v = \frac{C(N, s)}{2} |v|^{\frac{4s}{N-2s}} v$$

in a half-space. From [9, Theorem 1.1 and Remark 4.2], v is locally bounded (although the boundedness of a domain is assumed in [9], the proof works for our case). Then, in light of [4, Corollary 3] (see also [12, Corollary 1.6]), we know that the above problem in any half-space does not admit a nontrivial constant-sign solution. So v must be sign-changing, but then by a similar proof of Lemma 2.5, we have $\|v\|^2 \geq 2\mathbb{S}^{N/(2s)}$, which contradicts (2.12). So we find that $\text{dist}(x_n, \partial\Omega)/r_n \rightarrow \infty$. Then we can see that v is a nontrivial solution of (2.11). Using (2.12) again, we find that v is constant-sign and u is trivial. Setting

$$w_n(x) = u_n(x) - r_n^{(2s-N)/2} v((x - x_n)/r_n),$$

we have

$$I'(w_n) \rightarrow 0, \quad I(w_n) \rightarrow c - I(v) \quad \text{and} \quad \|w_n\|^2 \rightarrow \frac{Nc}{s} - \mathbb{S}^{N/(2s)} < \mathbb{S}^{N/(2s)}.$$

If $\|w_n\|_{L^{2N/(N-2s)}} \not\rightarrow 0$, repeating the argument above, we can obtain a contradiction. Hence, we have $\|w_n\| \rightarrow 0$ and $c = (s/N)\mathbb{S}^{N/(2s)}$. Therefore, we have shown our assertion. \square

We define $\mathcal{V}_0, \mathcal{Z}_0 : X_0 \setminus \{0\} \rightarrow X_0$ by

$$\mathcal{V}_0(u) = \frac{\nabla \mathcal{N}_0(u)}{\|\nabla \mathcal{N}_0(u)\|}, \quad \mathcal{Z}_0(u) = \nabla \mathcal{R}_0(u) - \langle \nabla \mathcal{R}_0(u), \mathcal{V}_0(u) \rangle \mathcal{V}_0(u) \quad \text{for each } u \in X_0 \setminus \{0\}.$$

Here, $\nabla \mathcal{N}_0(u)$ and $\nabla \mathcal{R}_0(u)$ are the elements of X_0 respectively obtained from $\mathcal{N}_0'(u)$ and $\mathcal{R}_0'(u)$ by the Riesz representation theorem. We note that

$$(2.13) \quad \langle \mathcal{Z}_0(u), \nabla \mathcal{N}_0(u) \rangle = 0 \quad \text{and} \quad \langle \mathcal{Z}_0(u), \nabla \mathcal{R}_0(u) \rangle = \|\mathcal{Z}_0(u)\|^2 \quad \text{for each } u \in \mathcal{M}.$$

The next proposition essentially says that $\mathcal{R}_0|_{\mathcal{M}}$ satisfies the Palais-Smale condition at any level in $(\mathbb{S}, 2^{2s/N}\mathbb{S})$. In the last section, we give a negative gradient flow of $\mathcal{R}_0|_{\mathcal{M}}$; see (3.2).

Proposition 2.7. *Let $\{v_n\}_n \subset \mathcal{M}$ which satisfies $\mathcal{Z}_0(v_n) \rightarrow 0$ in X_0 and $\mathcal{R}_0(v_n) \rightarrow c \in (\mathbb{S}, 2^{2s/N}\mathbb{S})$. Then $\{v_n\}_n$ has a convergent subsequence.*

Proof. For each $u \in \mathcal{M}$, we have

$$(2.14) \quad \begin{aligned} \|\mathcal{Z}_0(u)\|^2 &= \|\nabla \mathcal{R}_0(u) - \langle \nabla \mathcal{R}_0(u), \mathcal{V}_0(u) \rangle \mathcal{V}_0(u)\|^2 = \|\mathcal{R}_0(u)\|^2 - \langle \mathcal{R}_0(u), \mathcal{V}_0(u) \rangle^2 \\ &\geq \|\mathcal{R}_0(u)\|^2 \frac{\langle \mathcal{V}_0(u), u \rangle^2}{\|u\|^2} = \frac{2\|\mathcal{R}_0(u)\|^2}{\|u\|^2 \|\nabla \mathcal{N}_0(u)\|^2}. \end{aligned}$$

From our assumptions, we can infer that $\nabla \mathcal{R}_0(v_n) \rightarrow 0$. By virtue of Lemma 2.4, the sequence $u_n = \lambda_n v_n$, where λ_n is defined as in (2.9), is a Palais-Smale sequence for I_0 at level $(s/N)c^{N/(2s)}$. According to Proposition 2.6, there exists a subsequence of $\{u_n\}_n$ which converges strongly in X_0 . Since we have $\lambda_n \rightarrow c^{(N-2s)/(4s)}$ from Lemma 2.4, we can see that our assertion holds. \square

For the reader's convenience, we give the following lemma.

Lemma 2.8. *Let $\eta > 0$ and $u \in \mathcal{M}$ with $\mathcal{R}_0(u) \leq \mathbb{S} + \eta$. Then there exists $v \in \mathcal{M}$ such that $\|u - v\| \leq \sqrt{\eta}$, $\mathcal{R}_0(v) \leq \mathcal{R}_0(u)$ and $\|\mathcal{R}_0'(v)\| \leq \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}})$.*

Proof. By Ekeland's variational principle, we can find $v \in \mathcal{M}$ such that $\|u - v\| \leq \sqrt{\eta}$, $\mathcal{R}_0(v) \leq \mathcal{R}_0(u)$ and $\mathcal{R}_0(w) \geq \mathcal{R}_0(v) - \sqrt{\eta}\|w - v\|$ for each $w \in \mathcal{M}$. Fix $z \in X_0$ with $\|z\| = 1$. For each $s \in \mathbb{R}$ with $v + sz \neq 0$, there exists unique $t(s) > 0$ satisfying $t(s)(v + sz) \in \mathcal{M}$. Then we can easily see

$$t'(0) = - \int_{\Omega} |v|^{\frac{4s}{N-2s}} v z \, dx.$$

From

$$\mathcal{R}_0(v + sz) - \mathcal{R}_0(v) = \mathcal{R}_0(t(s)(v + sz)) - \mathcal{R}_0(v) \geq -\sqrt{\eta}\|t(s)(v + sz) - t(s)v + t(s)v - v\|,$$

we obtain

$$|\mathcal{R}'_0(v)(z)| \leq \sqrt{\eta} \|z + t'(0)v\| \leq \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}}),$$

which yields $\|\mathcal{R}'_0(v)\| \leq \sqrt{\eta}(1 + 1/\sqrt{\mathbb{S}})$. \square

3. PROOF OF THEOREM 1.1 CONCLUDED

In the following proof, we will repeatedly use the fact that $\mathcal{R}(\sigma u) = \mathcal{R}(u)$ for every $\sigma > 0$ and every $u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$. We write, for $u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$,

$$\Pi(u) = \frac{u}{\|u\|_{L^{2N/(N-2s)}}}.$$

From Propositions 2.1 and 2.2, we can find $C_3 > 0$ with

$$\mathcal{R}(\Pi(u_{\delta,\varepsilon,z})) \leq \frac{\|U_{1,0}\|^2 + C_1 \varepsilon^{N-2s}}{\left(\int_{\mathbb{R}^N} |U_{1,0}|^{\frac{2N}{N-2s}} dx - C_2 \varepsilon^N\right)^{\frac{N-2s}{N}}} \leq \mathcal{R}(U_{1,0}) + C_3 \varepsilon^{N-2s}$$

for each $\varepsilon \in (0, 1/20]$, $\delta \in (0, \varepsilon^2]$ and $z \in B_1$. Hence, we can find $\bar{\varepsilon} \in (0, 1/20]$ such that

$$\mathcal{R}(\Pi(u_{\bar{\varepsilon}^2, \bar{\varepsilon}, z})) \leq \varpi 2^{2s/N} \mathbb{S} \quad \text{for each } z \in B_1,$$

where $2^{-\frac{2s}{N}} < \varpi < 1$. Now, we fix $\delta = \bar{\varepsilon}^2$ and we define a kind of barycenter mapping

$$\beta(u) = \int_{\mathbb{R}^N} 1_{B_K}(x) x |u(x)|^{\frac{2N}{N-2s}} dx \quad \text{for each } u \in \dot{H}^s(\mathbb{R}^N) \text{ with } \|u\|_{L^{2N/(N-2s)}} = 1,$$

where $K = \sup\{|x| : x \in \Omega\} + 1$ and 1_{B_K} is the characteristic function for B_K . We also define

$$\bar{c} = \inf \{\mathcal{R}_0(u) : u \in \mathcal{M}, \beta(u) = 0\}.$$

Then, $\bar{c} > \mathbb{S}$. If not, there is a sequence $\{v_n\}_n \subset \mathcal{M}$ such that $\beta(v_n) = 0$ and $\mathcal{R}_0(v_n) \rightarrow \mathbb{S}$. From Lemma 2.8, we have $\mathcal{R}'_0(v_n) \rightarrow 0$. Then by Proposition 2.6, taking a subsequence if necessary, there exist $\{\lambda_n\}_n \subset (0, 1)$ and $\{z_n\}_n \subset \Omega$ such that $\lambda_n \rightarrow 0$, $z_n \rightarrow z \in \bar{\Omega}$ and

$$\text{either } \|v_n - \Pi(U_{\lambda_n, z_n})\| = o(1) \quad \text{or} \quad \|v_n + \Pi(U_{\lambda_n, z_n})\| = o(1) \quad \text{as } n \rightarrow \infty.$$

From $\beta(v_n) = 0$ and $\beta(v_n) \rightarrow z$, we obtain $0 \in \bar{\Omega}$, which is a contradiction. Now, from Propositions 2.1 and 2.2, we can find a map $f: B_1 \rightarrow \mathcal{M}$ which satisfies

$$\mathcal{R}_0(f(z)) \leq \varpi 2^{2s/N} \mathbb{S} \quad \text{for each } z \in B_1,$$

$$\mathcal{R}_0(f(z)) \leq \frac{\mathbb{S} + \bar{c}}{2} < \bar{c} \quad \text{for each } z \in \partial B_1$$

and

$$(3.1) \quad |\beta(f(z)) - z| \leq \frac{1}{2} \quad \text{for each } z \in \partial B_1.$$

Such f can be obtained by setting $f(z) = u_{\bar{\varepsilon}^2, h_\varepsilon(|z|), z}$ with sufficiently small $\varepsilon > 0$, where

$$h_\varepsilon(t) = \begin{cases} \bar{\varepsilon} & \text{for } 0 \leq t \leq 1/2, \\ 2(1-t)\bar{\varepsilon} + (2t-1)\varepsilon & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

and we can show (3.1) by a similar argument above which shows $\bar{c} > \mathbb{S}$. Then, for each $t \in [0, 1]$ and $z \in \partial B_1$, we have $|(1-t)z + t\beta(f(z))| \geq |z| - t|\beta(f(z)) - z| \geq 1/2$. So by using Brouwer's degree theory, we have $\deg(\beta \circ f, \text{Int}(B_1), 0) = 1$. Defining

$$c = \inf_{g \in G} \max_{x \in B_1} \mathcal{R}_0(g(x)), \quad G = \{g \in C(B_1, \mathcal{M}) : g = f \text{ on } \partial B_1 \text{ and } \deg(\beta \circ g, \text{Int}(B_1), 0) = 1\},$$

we have

$$\mathbb{S} < \bar{c} \leq c \leq \varpi 2^{2s/N} \mathbb{S}.$$

Now, we will show there is $u \in \mathcal{M}$ such that $\nabla \mathcal{R}_0(u) = 0$ and $\mathcal{R}_0(u) = c$. Assume not. By Proposition 2.7, we can choose a positive constant $\eta > 0$ such that $(\mathbb{S} + c)/2 < c - 2\eta$, $c + 2\eta < \varpi 2^{2s/N} \mathbb{S}$ and $\mathcal{L}_0(u) \neq 0$ for each $u \in \mathcal{M}$ with $|\mathcal{R}_0(u) - c| \leq 3\eta$. We also choose a locally Lipschitz function $\alpha : \mathcal{M} \rightarrow [0, 1]$ such that

$$\alpha(u) = \begin{cases} 1 & \text{for each } u \in \mathcal{M} \text{ with } |\mathcal{R}_0(u) - c| \leq \eta, \\ 0 & \text{for each } u \in \mathcal{M} \text{ with } |\mathcal{R}_0(u) - c| \geq 2\eta. \end{cases}$$

Then we can define $\gamma : [0, 1] \times \mathcal{M} \rightarrow \mathcal{M}$ by

$$(3.2) \quad \gamma(0, u) = u \quad \text{and} \quad \frac{d}{dt}\gamma(t, u) = -\frac{2\eta\alpha(\gamma(t, u))}{\|\mathcal{Z}_0(\gamma(t, u))\|^2} \mathcal{Z}_0(\gamma(t, u));$$

see (2.13) and (2.14). Let $g \in G$ such that $\max_{z \in B_1} \mathcal{R}_0(g(z)) < c + \eta$. Then we can easily see $\gamma(t, g(z)) = g(z)$ for each $(t, z) \in [0, 1] \times \partial B_1$, which yields $\deg(\beta(\gamma(1, g(\cdot))), \text{Int}(B_1), 0) = 1$. Moreover, we can find $\mathcal{R}_0(\gamma(1, g(z))) \leq c - \eta$ for each $z \in B_1$, which contradicts the definition of c . From Proposition 2.6, we can find that this contradiction proves the existence of a nonnegative weak solution to (2.2). By [13, Theorem 2.5], the obtained solution is positive in Ω . \square

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